Journal of Sound and Vibration (1997) **199**(1), 87–106

FUNDAMENTAL FREQUENCY DETERMINATION OF STIFFENED PLATES USING SEQUENTIAL QUADRATIC PROGRAMMING

O. K. Bedair

Civil Engineering Department, Concordia University, 1455 de Maisonneuve Blvd. W., Montreal, Quebec, H3G 1M8, Canada

(*Received* 20 *October* 1995, *and in final form* 17 *June* 1996)

A novel approach for the determination of the fundamental frequency of stiffened plates is presented. As a first stage, the governing differential equations for the structure are derived. Then, an energy formulation is presented in which the structure is idealized as assembled plate and stiffener elements, rigidly connected at their junctions. The non-linear strain energy function of the assembled structure is then transformed into an unconstrained optimization problem and Sequential Quadratic Programming (SQP) is used to determine the magnitudes of the lowest natural frequency and the associated mode shape. Using the described algorithm, results are presented showing the variation of the natural frequency with plate/stiffener geometric parameters for various concentric and eccentric stiffening configurations.

1. INTRODUCTION

Interest in stiffened plate constructions has been wide spread in recent years in order to achieve an economical, lightweight design of the structure. While stiffening elements add small weight to the overall structure their influence on the total stiffness is enormous. Stiffened plate construction is widely used in aerospace and marine structures, where weight is of great significance, and in engineering structures such as bridge decks, box and plate girders, . . . etc.

Introducing stiffeners to the plate complicates the analysis and several assumptions must be made in order to facilitate a solution to the problem. The complication increases if the stiffeners have different cross-sectional properties or are unequally spaced. Various methods has been presented for the free vibration analysis of stiffened plates. Wah [1] presented a procedure for the analysis of equally spaced, concentric stiffeners with identical cross-sectional properties. Asku and Ali [2, 3] presented an alternative numerical procedure for equally spaced stiffeners. The method is based on variational principles in conjunction with finite difference techniques to determine the natural frequency of the structure. They illustrated the method by the analysis of a plate with one longitudinal stiffener and a plate with one longitudinal and one transverse stiffener. Mukhopadhyay [4, 5] presented a semi-analytical finite difference procedure for the dynamic analysis of stiffened plates. The governing differential equations of the structure are first developed by assuming the stiffeners to be symmetric about the mid-plane of the plate and ignoring the torsional stiffness of the stiffeners. A displacement function satisfying the boundary condition is then substituted into the governing differential equations and the resulting equations are transformed into ordinary differential equations with constant coefficients that are solved

0022–460X/97/010087 + 20 $25.00/0/sv960606

by a finite difference scheme. Other approximate methods have also been used by other researchers for the analysis of such structures [e.g. Kirk [11, 12], Long [13, 14], Mead *et al*. [15]]. The Finite Element [6–9] and the Finite Strip Methods [10] have also been used. Some finite element formulations are based on the orthotropic plate model and others treat it as a discrete model. The plate in this case is divided into subelements and the stiffeners are treated as beam or plate elements with imposed compatibility conditions along the line of junctions.

In this paper, an alternative numerical procedure for the free vibration analysis of multi-stiffened plates is presented. The method takes into account the discrete nature of the structure and is flexible for handling non-uniform stiffening systems. The objectives of most researchers of this subject have been simply to analyze the structure rather than study its behaviour. An objective of this paper is also to study the influence of plate/stiffener proportions on the free vibration characteristics. This part of the investigation will provide valuable information for design purposes.

2. THEORETICAL ANALYSIS

Consider the plate shown in Figure 1, of length a and width b, stiffened orthogonally by NS^x eccentric stiffeners parallel to the y-axis and NS^y stiffeners parallel to the x-axis. The origin of the global axis of the plate is chosen at the lower left hand corner denoted by O. Note also that ξ and η are non-dimensional parameters x/a and y/b, respectively. The distance from the global co-ordinate system to the centroid of stiffeners along the y-axis is denoted by η_i and for stiffeners along the x-axis is denoted by ξ_i as shown in the figure. The spacing of the stiffeners along the x-direction is also denoted by sp^{xi}, $i = 1, 2, 3, \ldots, NS^x$, and for stiffeners along the y-axis, is denoted by sp^{yi}, $i = 1, 2, 3, \ldots, NS^y$. The first letter of the superscript denotes the axis along which the stiffeners span and the second denotes the span number from the origin O. Each stiffener is also described by a set of geometric properties, cross-sectional area A, first moment of

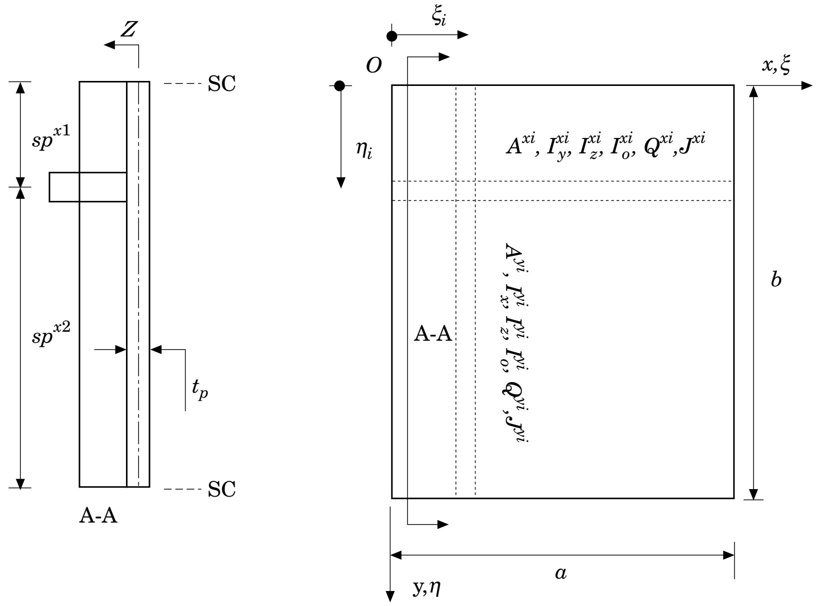

Figure 1. Orthogonally stiffened plate.

inertia Q, second moments of inertia about the major and minor axes i.e. I_y, I_z for stiffeners along the x-axis and I_x, I_z for stiffeners along the y-axis, polar moment of inertia I_0 and torsional rigidity, J. Therefore, stiffeners along the x-axis will be identified by superscript xi and stiffeners along the y-axis by yi, where the letter i represents the stiffener number. As an illustration of this labelling scheme, if the plate is stiffened by three identical equally spaced stiffeners along the x-axis and two identical equally spaced stiffeners along the y-axis, then $sp^{x1} = sp^{x2} = sp^{x3} = sp^{x4} = b/4$, $sp^{y1} = sp^{y2} = sp^{y3} = a/3$ and $I_x^{x1} = I_x^{x2} = \ldots I_x^{x3}, \ldots$ etc. In the present formulation, the geometric properties of the plate will be labelled by the subscript or superscript p, e.g., thickness of the plate t_p, modulus of elasticity E_p, and the modulus of elasticity for the stiffeners will be denoted E_{st}.

The subsequent sections describe two approaches to the free vibration analysis of stiffened plates. The plate and stiffeners for both cases are treated as assembled plate and beam elements. In the first approach, the governing differential equations are derived and, in the second, an energy formulation, which will be used in subsequent sections as a method of analysis, will be presented. The derivations are based on the following assumptions: (1) the plate and stiffeners are made of isotropic, perfectly elastic materials; (2) the stiffeners are rigidly connected to the plate which implies no relative rotation at their junctions; (3) the stiffeners have open cross-section; (4) thin plate theory applies for the plate; (5) the axial strain of the stiffeners is not affected by the restraint imposed by the perpendicular stiffeners at their points of intersection; (6) the torsional rigidity of the stiffeners can be estimated by St. Venant's theory.

2.1. GOVERNING DIFFERENTIAL EQUATIONS

The derivation in this section is restricted to identical and equally spaced stiffeners. By taking the midplane of the plate as the axis of reference, the resultant forces for the assembled structure are given on p. 4, where $A_p = t_p/(1 - \nu^2)$, and Q^{yi} denotes the first moment of area of the stiffeners per unit width about the mid-plane of the plate and A^{xi} and A^{yi} are the cross-sectional areas of typical x-wise and y-wise stiffeners, respectively. A list of symbols appears in the Appendix.

Note that the displacement functions W, U, and V for the plate are two-dimensional, while for the stiffeners they are one-dimensional. Since the same functions are used for both elements, the transformation delta functions, $\delta(\eta - \eta_i)$, $\delta(\xi - \xi_i)$, have been introduced into equation (1) which evaluates the displacement function at the location of the stiffener. For example, when this transformation function is used with the out-of-plane displacement function, then

$$W(\xi, \eta)\,\delta(\eta - \eta_i) = W(\xi, \eta_i), \tag{2}$$

where η_i is the location of the stiffener spanning along the x- or ξ-axis.

The moment resultants are given by equation (3) on p. 5, where I_p is the plate flexural rigidity per unit width $= t_p^3/12(1 - \nu^2)$; the product $E_p I_p$ in the conventional plate flexural rigidity, D, is so designated since the plate and stiffeners might have different moduli of elasticity.

Substituting equations (1) into the force equilibrium equations results in the following pair of differential equations:

$$\beta^{-1}\left[A_p + \frac{E_{st}}{E_p} A^{xi}\,\delta(\eta - \eta_i)\right]\frac{\partial^2 U}{\partial \xi^2} + \frac{1}{2}(1 + \nu)A_p \frac{\partial^2 V}{\partial \xi \partial \eta} + \frac{\beta}{2}(1 - \nu)A_p \frac{\partial^2 U}{\partial \eta^2}$$

$$- \frac{\beta^{-1}}{a}\frac{E_{st}}{E_p} Q^{xi}\,\delta(\eta - \eta_i)\frac{\partial^3 W}{\partial \xi^3} = [m_p + m_{st}^{xi}\,\delta(\eta - \eta_i) + m_{st}^{yi}\,\delta(\xi - \xi_i)\frac{\partial^2 U}{\partial t^2}, \tag{4}$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = E_p \begin{bmatrix} \left(A_p + \frac{E_{st}}{E_p} A^{xi}\,\delta(\eta - \eta_i)\right) & vA_p & -\frac{E_{st}}{E_p} Q^{xi}\,\delta(\eta - \eta_i) & 0 & 0 \\ vA_p & \left(A_p + \frac{E_{st}}{F_p} A^{yi}\,\delta(\xi - \xi_i)\right) & 0 & -\frac{E_{st}}{E_p} Q^{yi}\,\delta(\xi - \xi_i) & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1 - v)A_p \end{bmatrix} \begin{Bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial^2 W}{\partial x^2} \\ \frac{\partial^2 W}{\partial y^2} \\ \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \end{Bmatrix}, \quad (1)$$

$$
\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = E_p \begin{bmatrix} -\frac{E_{st}}{E_p} Q^{xi}\,\delta(\eta-\eta_i) & 0 & \left(I_p + \frac{E_{st}}{E_p} I_y^{xi}\,\delta(\eta-\eta_i)\right) & \nu I_p & 0 \\ 0 & -\frac{E_{st}}{E_p} Q^{yi}\,\delta(\xi-\xi_i) & \nu I_p & \left(I_p + \frac{E_{st}}{E_p} I_x^{yi}\,\delta(\xi-\xi_i)\right) & 0 \\ 0 & 0 & 0 & 0 & 2(1-\nu) I_p + \frac{G_{st}}{2E_p}\left(J^{xi}\delta(\eta-\eta_i) + J^{yi}\delta(\xi-\xi_i)\right) \end{bmatrix} \times \begin{Bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial^2 W}{\partial x^2} \\ \frac{\partial^2 W}{\partial y^2} \\ \frac{\partial^2 W}{\partial x\,\partial y} \end{Bmatrix}, \tag{3}
$$

$$\beta\left[A_p + \frac{E_{st}}{E_p} A^{yi}\,\delta(\xi - \xi_i)\right]\frac{\partial^2 V}{\partial \eta^2} + \frac{1}{2}(1+v)A_p\frac{\partial^2 U}{\partial \xi\,\partial \eta} + \frac{\beta^{-1}}{2}(1-v)A_p\frac{\partial^2 V}{\partial \xi^2}$$

$$-\frac{\beta}{b}\frac{E_{st}}{E_p} Q^{yi}\,\delta(\xi - \xi_i)\frac{\partial^3 W}{\partial \eta^3} = [m_p + m_{st}^{xi}\,\delta(\eta - \eta_i) + m_{st}^{yi}\,\delta(\xi - \xi_i)]\frac{\partial^2 V}{\partial t^2}, \qquad (5)$$

where m^p and m_{st}^{xi} and m_{st}^{yi} are the mass density of the plate and stiffeners along the x- and y-axes, respectively, ξ and η are non-dimensional parameters x/a and y/b, respectively, and β is the plate aspect ratio, a/b.

Similarly, by substituting equation (2) the moment equilibrium equation results in an additional differential equation

$$\left[I_p + \frac{E_{st}}{E_p} I_y^{xi}\,\delta(\eta - \eta_i)\right]\frac{\partial^4 W}{\partial \xi^4} + 2\beta^2\left[I_p + \frac{G_{st}}{2E_p}(J^{xi}\,\delta(\eta - \eta_i) + J^{yi}\,\delta(\xi - \xi_i))\right]\frac{\partial^4 W}{\partial \xi^2\,\partial \eta^2}$$

$$+\beta^4\left[I_p + \frac{E_{st}}{E_p} I_x^{yi}\,\delta(\xi - \xi_i)\right]\frac{\partial^4 W}{\partial \eta^4} - a\frac{E_{st}}{E_p}\left[Q^{xi}\,\delta(\eta - \eta_i)\frac{\partial^3 U}{\partial \xi^3} + \beta^3 Q^{yi}\,\delta(\xi - \xi_i)\frac{\partial^3 V}{\partial \eta^3}\right]$$

$$= -a^2\frac{m_p}{E_p}\left[a^2\frac{\partial^2 W}{\partial t^2} + \frac{m_{st}^{xi}}{m_p}\,\delta(\eta - \eta_i)\left(a^2\frac{\partial^2 W}{\partial t^2} + I_0^{xi}\beta^2\frac{\partial^4 W}{\partial \eta^2 t^2}\right)\right.$$

$$\left.+\frac{m_{st}^{yi}}{m_p}\,\delta(\xi - \xi_i)\left(a^2\frac{\partial^2 W}{\partial t^2} + I_0^{yi}\frac{\partial^4 W}{\partial \xi^2 t^2}\right)\right], \qquad (6)$$

where I_0^{xi} and I_0^{yi} are the polar moments of inertia of x-wise and y-wise stiffeners, respectively. Therefore, for given plate and stiffener properties, the solution of the three coupled differential equations (4–6) gives the natural frequency for the assembly.

2.2. ENERGY FORMULATION

The energy method affords an alternative means of analyzing stiffened plates. In dealing with the structure as assembled plate and beam (or stiffener) elements, the strain energy, in the interval $-t_p/2 < z < t_p/2$, is given by

$$U_p = \frac{1}{2}\int\int_V\int \sigma_{ij}^p \epsilon_{ij}^p\,\mathrm{d}V \qquad (7)$$

and the strain energy of the longitudinal and transverse stiffeners is composed of two parts, the axial and shear strain components, i.e.,

$$U_{st}^{xi} = \frac{E_{st}}{2}\int\int_V\int (\epsilon_{\mathrm{xx}}^{\mathrm{xi}})^2\,\mathrm{d}V + \frac{(GJ^{xi})}{2a}\int_0^1\left(\frac{\partial \theta}{\partial \xi}\right)^2\mathrm{d}\xi, \qquad (8)$$

$$U_{st}^{yi} = \frac{E_{st}}{2}\int\int_V\int (\epsilon_{yy}^{yi})^2\,\mathrm{d}V + \frac{(GJ^{yi})}{2b}\int_0^1\left(\frac{\partial \theta}{\partial \eta}\right)^2\mathrm{d}\eta, \qquad (9)$$

where U_{st}^{xi} and U_{st}^{yi} are the strain energies of the x-wise and y-wise stiffeners, respectively.

Considering a typical ith stiffener along the x-axis, the axial strain is given by

$$\epsilon_{xx}^{xi} = (\epsilon_{xx}^p)_{z=t_p/2} - (z - t_p/2)(\mathrm{d}^2 W/\mathrm{d}x^2) - y_i(\mathrm{d}^2 V/\mathrm{d}x^2) + c_w(\mathrm{d}^2\theta/\mathrm{d}x^2), \qquad (10)$$

where c_w is the warping constant of the stiffener and θ is the angle of rotation. The first component is the plate strain evaluated at $z = t_p/2$, the second and third components are the axial strains due to bending about the major and minor axes of their local co-ordinates, and the last component is the axial strain due to warping. The plate kinetic energy is given by

$$T_p = \frac{m_p \omega^2}{2} ab \int_0^1 \int_0^1 \left(\frac{\partial W}{\partial t} \right) \mathrm{d}\xi \, \mathrm{d}\eta, \tag{11}$$

where m_p is the mass density of the plate and ω is the natural frequency of the stiffened plate.

The kinetic energy of a typical stiffener along the x-axis is given by

$$T_{st}^{xi} = \frac{m_{st}^{si} \omega^2}{2} A^{xi} a \int_0^1 \left(\frac{\partial W}{\partial t} \right)^2 \mathrm{d}\xi + \frac{m_{st}^{xi} \omega^2}{2b} I_0^{xi} \beta \int_0^1 \left(\frac{\partial^2 W}{\partial t \, \partial \eta} \right)^2 \mathrm{d}\xi. \tag{12}$$

Similarly, the kinetic energy of a typical stiffener along the y-axis is given by

$$T_{st}^{yi} = \frac{m_{st}^{yi} \omega^2}{2} A^{yi} b \int_0^1 \left(\frac{\partial W}{\partial t} \right)^2 \mathrm{d}\eta + \frac{m_{st}^{yi} \omega^2}{2a} I_0^{yi} \beta^{-1} \int_0^1 \left(\frac{\partial^2 W}{\partial t \, \partial \xi} \right)^2 \mathrm{d}\eta. \tag{13}$$

Equating the maximum kinetic and strain energies, the natural frequency of the assembled structure, ω, can be expressed as

$$\omega = (1/a^2) 4 \sqrt{D/m_p t_p} \, \Omega. \tag{14}$$

where Ω is a natural frequency parameter that is a function of the plate/stiffener geometric properties and the displacement functions W, U and V.

2.3. SEQUENTIAL QUADRATIC PROGRAMMING

The total potential of the assembled structure is composed of the strain energy of the plate, U_p, the strain energies of the stiffeners in the x- and y-directions, U_{st}^{xi}, U_{st}^{yi} and the kinetic energy of the plate and stiffeners. The out-of- and in-plane displacements or shape functions, $W(\xi, \eta)$, $U(\xi, \eta)$ and $V(\xi, \eta)$, can be expressed as

$$W(\xi, \eta) = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} w_{ij} F_i(\xi) G_j(\eta), \qquad U(\xi, \eta) = \sum_{m=1}^{M_2} \sum_{n=1}^{N_2} u_{mn} B_m(\xi) D_n(\eta), \tag{15, 16}$$

$$V(\xi, \eta) = \sum_{r=1}^{M_3} \sum_{s=1}^{N_3} v_{rs} E_r(\xi) H_s(\eta), \tag{17}$$

where $F_i(\xi)$ and $G_j(\eta)$ are generalized functions that may be polynomials, harmonics $\cdots$ etc., satisfying out-of-plane boundary conditions at $\xi = 0$, 1 and $\eta = 0$, 1, respectively, and w_{ij} are associated coefficients for the $F_i(\xi)$ and $G_j(\eta)$ functions. Similarly $B_m(\xi)$, $D_n(\eta)$, $E_r(\xi)$ and $H_s(\eta)$ are generalized functions that satisfy the in-plane boundary conditions and $\{u_{mn}, v_{rs}\}$ are their corresponding amplitudes. The integers N_1, M_1, N_2, M_2, N_3, M_3 denote the number of generalized functions used to define the displacement functions $W(\xi, \eta)$, $U(\xi, \eta)$ and $V(\xi, \eta)$.

The objective now is to find, for prescribed $F_i(\xi)$, $G_j(\eta)$, $B_m(\xi)$, $D_n(\eta)$, $E_r(\xi)$ and $H_s(\eta)$, the coefficients $\{w_{ij}, u_{mn}, v_{rs}\}$ that minimize the parameter Ω. Since this parameter is a non-linear function of these coefficients, analytical treatment of the problem becomes difficult especially as the number of coefficients increases. In this investigation the Sequential Quadratic Programming (SQP) algorithm has been used as the optimization algorithm for the vibration analysis of stiffened plates. Various versions of the algorithm can also be bound in reference [16].

The mathematical statement of the problem is

$$\textit{minimize} \quad \Omega(w_{ij}, u_{mn}, v_{rs}), \qquad \textit{subject to} \quad \begin{bmatrix} \{w_{ij}\} \\ \{u_{mn}\} \\ \{v_{rs}\} \end{bmatrix}^L \leqslant \begin{bmatrix} \{w_{ij}\} \\ \{u_{mn}\} \\ \{v_{rs}\} \end{bmatrix} \leqslant \begin{bmatrix} \{w_{ij}\} \\ \{u_{mn}\} \\ \{v_{rs}\} \end{bmatrix}^U, \quad (18, 19)$$

where superscripts U and L denote the upper and lower bounds on these variables.

The optimization strategy of SQP for the non-linear function is performed iteratively by generating and solving a sequence of quadratic sub-problems. The optimization strategy is described by

$$\{\mathbf{x}_i^{k+1}\} = \{x_i^k\} + \{\alpha^k\}\{P^k\}, \quad (20)$$

where the superscript denotes the iteration number and the subscript i denotes the design variable, $\mathbf{x}$ is the vector containing the displacement coefficients, $\{w_{ij}, u_{mn}, v_{rs}\}$, α is the step size and P is the search direction.

The search direction, P, is computed in the SQP algorithm from the solution of the quadratic, Taylor expansion of the frequency parameter Ω:

$$\textit{minimize} \quad \nabla\Omega(w_{ij}, u_{mn}, v_{rs}) \cdot P + \tfrac{1}{2}P^{\mathrm{T}} \cdot \nabla^2\mathbf{\Omega}(w_{ij}, u_{mn}, v_{rs}) \cdot P, \quad (21)$$

where $\nabla\Omega = \partial\Omega/\partial x_i$ is the gradient of the natural frequency parameter at the kth iteration and $\nabla^2\mathbf{\Omega}$ is the matrix of the second derivative (or the Hessian matrix):

$$\nabla^2\mathbf{\Omega} = \begin{bmatrix} \dfrac{\partial^2\Omega}{\partial x_1^2} & \dfrac{\partial^2\Omega}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2\Omega}{\partial x_1\,\partial x_j} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2\Omega}{\partial x_i\,\partial x_1} & \dfrac{\partial^2\Omega}{\partial x_i\,\partial x_2} & \cdots & \dfrac{\partial^2\Omega}{\partial x_i\,\partial x_j} \end{bmatrix}. \quad (22)$$

After obtaining the search direction from this quadratic approximation of function (21), each iteration proceeds by determining a step length, α, that produces a sufficient decrease in the natural frequency function Ω.

The minimization process is illustrated schematically in Figure 2. The process starts, at point A, by selecting initial values for the variables $x^0 = \{w_{ij}, u_{mn}, v_{rs}\}^0$. A search direction, P^0, from equation (21) and step size α^0 are then computed to determine a new set of variables $\{w_{ij}, u_{mn}, v_{rs}\}^{\mathrm{B}}$ that lower the natural frequency function, Ω, to point B, in other words

$$\Omega(\{w_{ij}\}, \{u_{mn}\}, \{v_{rs}\}^{\mathrm{B}} < \Omega(\{w_{ij}\}, \{u_{mn}\}, \{v_{rs}\})^{\mathrm{A}}, \quad (23)$$

where Ω^{A} and Ω^{B} are the values of the functions at points A and B. The process continues until there is no further decrease in Ω, or the decrease is of negligible order, i.e.,

$$|\Omega(\{w_{ij}\}, \{u_{mn}\}, \{v_{rs}\})^{k-1} - \Omega(\{w_{ij}\}, \{u_{mn}\}, \{v_{rs}\})^k| \leqslant \epsilon, \quad (24)$$

where k is the iteration number and ϵ is a small parameter number, chosen in this study as 0·01.

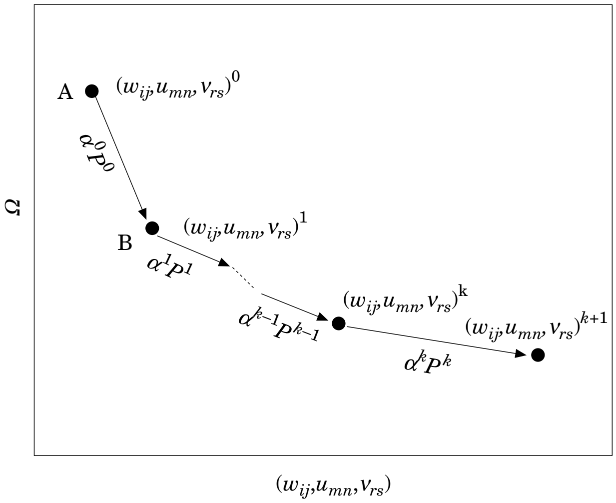

Figure 2. Schematic of minimization procedure of SQP.

3. RESULTS AND DISCUSSION

For a plate simply supported along four edges, admissible displacement functions satisfying zero out of plane deflection, W, and bending moments along the four edges can be assumed as

$$F_i(\xi) = \sin(i\pi\xi), \qquad G_j(\eta) = \sin(j\pi\eta), \qquad B_1(\xi) = (1/2 - \xi),$$
$$B_{m+1}(\xi) = \sin(m\pi\xi), \qquad D_1(\eta) = 1, \tag{25}$$

$$D_{n+1}(\eta) = \sin(n\pi\eta), \qquad E_1(\eta) = 1, \qquad E_{r+1}(\xi) = \sin(r\pi\xi),$$
$$H_1(\eta) = (1/2 - \eta), \qquad H_{s+1}(\eta) = \sin(s\pi\eta). \tag{26}$$

When using equations (25, 26), the natural frequency parameter, Ω, of equation (14) becomes equation (27) on p. 10, where

$$P_1 = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \{i^4, \beta^4 j^4, 2i^2 j^2\} \begin{Bmatrix} w_{ij}^2 \\ w_{ij}^2 \\ w_{ij}^2 \end{Bmatrix}, \tag{28}$$

$$P_2 = \sum_{m=1}^{M_2} \sum_{n=1}^{N_2} \Bigg\{ (\beta\pi)^{-1}, \beta^{-1} m^2, \tfrac{1}{2}(1-v)\beta n^2, \frac{32}{\pi^2} v \frac{mnrsn}{(r^2-m^2)(n^2-s^2)} \alpha_{mr}\alpha_{ns}, \qquad \frac{16}{\pi^2}(1-v)$$
$$\frac{mnrs}{(m^2-r^2)(s^2-n^2)} \alpha_{mr}\alpha_{ns} \Bigg\} \begin{Bmatrix} u_0^2 \\ u_{mn}^2 \\ u_{mn}^2 \\ u_{mn}v_{rs} \\ u_{mn}v_{rs} \end{Bmatrix} + [(\beta\pi)^{-2}, \beta s^2, \tfrac{1}{2}(1-v)\beta^{-1} r^2, 2v\pi^{-2}] \begin{Bmatrix} v_0^2 \\ v_{rs}^2 \\ v_{rs}^2 \\ v_0 u_0 \end{Bmatrix} \tag{29}$$

$$\Omega = \pi^2 \left\{ \frac{\beta^{-2}P_1 + 0{\cdot}12\left(\frac{b}{t}\right)^2 P_2 + \sum_{i=1}^{Ns^x} [0{\cdot}2\gamma_1,\, 2\beta^{-2}\gamma_2,\, 0{\cdot}4\beta^{-1}\gamma_3,\, 2\gamma_4] \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} + \sum_{i=1}^{Ns^y} [0{\cdot}2\beta\gamma_5,\, 2\beta\gamma_6,\, 0{\cdot}4\beta\gamma_7,\, 2\beta^{-1}\gamma_8] \begin{bmatrix} I_5 \\ I_6 \\ I_7 \\ I_8 \end{bmatrix}}{\beta^{-2} \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} w_{ij}^2 + 2 \sum_{i=1}^{Ns^x} [\beta^{-2} I_9\, \delta_1^{xi} + \pi^2 \beta I_{10}\, \delta_2^{xi}] + 2 \sum_{i=1}^{Ns^y} [\beta^{-3} I_{11}\, \delta_1^{yi} + \pi^2 \beta^{-2} I_{12}\, \delta_2^{yi}]} \right\}^{1/2}, \tag{27}$$

$$I_1 = \frac{2}{\pi^2} u_0^2 + \sum_{m=1}^{M_2} \sum_{n=1}^{N_2} \sum_{q=1}^{N_2} m^2 u_{mn} u_{mq} \sin(n\pi\eta) \sin(q\pi\eta)\, \delta(\eta - \eta_i), \tag{30}$$

$$I_2 = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{l=1}^{N_1} i^4 \sin(j\pi\eta) \sin(l\pi\eta)\, \delta(\eta - \eta_i) w_{ij} w_{il}, \tag{31}$$

$$I_3 = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} 2i\pi \sin(j\pi\eta)\, \delta(\eta - \eta_i) u_0 w_{ij} + \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{m=1}^{M_2} \sum_{n=1}^{N_2} 2\pi \frac{i^3 m^2}{m^2 - i^2} \alpha_{mi}$$

$$\times \sin(n\pi\eta) \sin(j\pi\eta)\, \delta(\eta - \eta_i) u_{mn} w_{ij}, \tag{32}$$

$$I_4 = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{l=1}^{N_1} i^2 jl \cos(j\pi\eta) \cos(l\pi\eta)\, \delta(\eta - \eta_i) w_{ij} w_{il}, \tag{33}$$

$$I_5 = \frac{2}{\pi^2} v_0^2 + \sum_{r=1}^{M_3} \sum_{g=1}^{M_3} \sum_{s=1}^{N_3} s^2 v_{rs} v_{gs} \sin(r\pi\eta) \sin(g\pi\eta)\, \delta(\xi - \xi_i), \tag{34}$$

$$I_6 = \sum_{i=1}^{M_1} \sum_{k=1}^{M_1} \sum_{j=1}^{N_1} j^4 \sin(i\pi\xi) \sin(k\pi\xi)\, \delta(\xi - \xi_i) w_{ij} w_{kj}, \tag{35}$$

$$I_7 = \sum_{i=1}^{N_1} \sum_{j=1}^{N_1} 2j\pi \sin(i\pi\xi)\, \delta(\xi - \xi_i) v_0 w_{ij} + \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{r=1}^{M_3} \sum_{s=1}^{N_3} 2\pi \frac{j^3 s^2}{s^2 - j^2} \alpha_{sj}$$

$$\times \sin(i\pi\xi) \sin(r\pi\xi)\, \delta(\xi - \xi_i) v_{rs} w_{ij}, \tag{36}$$

$$I_8 = \sum_{i=1}^{M_1} \sum_{k=1}^{M_1} \sum_{j=1}^{N_1} j^2 ik \cos(i\pi\xi) \cos(k\pi\xi)\, \delta(\xi - \xi_i) w_{ij} w_{kj}, \tag{37}$$

$$I_9 = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{k=1}^{M_1} \sum_{l=1}^{N_1} \sin(j\pi\eta) \sin(l\pi\eta) w_{ij} w_{kl}\, \delta(\eta - \eta_i), \tag{38}$$

$$I_{10} = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{k=1}^{M_1} \sum_{l=1}^{N_1} \cos(j\pi\eta) \cos(l\pi\eta) w_{ij} w_{kl}\, \delta(\eta - \eta_i), \tag{39}$$

$$I_{11} = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{k=1}^{M_1} \sum_{i=1}^{N_1} \sin(i\pi\xi) \sin(k\pi\xi) w_{ij} w_{kl}\, \delta(\xi - \xi_i), \tag{40}$$

$$I_{12} = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \sum_{k=1}^{M_1} \sum_{l=1}^{N_1} \cos(j\pi\eta) \cos(l\pi\eta) w_{ij} w_{kl}\, \delta(\eta - \eta_i), \tag{41}$$

TABLE 1

Comparison of Ω obtained using SQP reference [17]

b/a	Ω: Reference [17]	Ω: SQP
1	19·74	19·73
1·5	14·26	14·17
2	12·34	12·33
2·5	11·45	11·43

where α_{mi} is a parameter which equals 0 if $m + i$ is even and 1 otherwise; γ_i represents the plate/stiffeners geometric proportions which depends upon the profile of the stiffeners. Their magnitudes are given by

$$\begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \delta_1^{xi} \\ \delta_2^{xi} \end{Bmatrix} = E_{st} \begin{Bmatrix} A^{xi}b/D \\ I_y^{xi}/Db \\ A^{xi}e^{xi}/D \\ (1/2(1+v))(J^{xi}/Db) \\ A^{xi}/E_{st}bt_p \\ I_0^{xi}/E_{st}a^3t_p \end{Bmatrix}, \qquad \begin{Bmatrix} \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \\ \delta_1^{yi} \\ \delta_2^{yi} \end{Bmatrix} = E_{st} \begin{Bmatrix} A^{yi}b/D \\ I_x^{yi}/Db \\ A^{yi}e^{yi}/D \\ (1/2(1+v))(J^{yi}/Db) \\ A^{yi}/E_{st}bt_p \\ I_0^{yi}/a^3t_p \end{Bmatrix}. \tag{42}$$

A computer program was developed using the energy formulation and the SQP technique, described previously for the analysis of multi-stiffened plates. With input data describing the plate-stiffener geometric proportions and the panel configuration, i.e., number of longitudinal and transverse stiffeners and stiffeners spacing, the lowest natural frequency and the associated mode shape are determined for the structure. The results of this section are grouped into two major parts.

In the first section, the program is verified for several unstiffened and stiffened panels that have been analyzed by other authors, using alternative numerical methods, such as finite element, finite difference and finite strip. In the second part, the effect of the plate/stiffener proportion on the natural frequency of the structure is investigated for several panels.

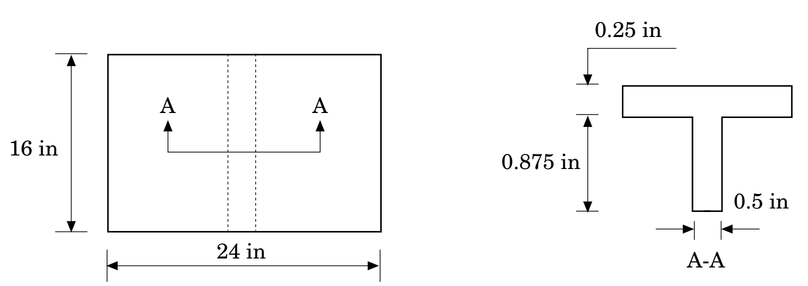

Figure 3. Geometric details of plate with one longitudinal stiffener.

TABLE 2

Comparison of Ω obtained using SQP and references [3, 8, 7]

Figure	Reference	ω (Hz)
3	3	254·94
	8	253·6
	7	257·05
	Present	256·2

3.1. VERIFICATION EXAMPLES

3.1.1. *Example* 1

As a first stage, the program was verified for unstiffened simply supported plates. The numerical values obtained using SQP are compared with the analytical values of reference [17] in Table 1, for four aspect ratios, $b/a = 1$, 1·5, 2, 2·5. The values obtained using SQP were based on 16 out-of-plane displacement coefficients. The natural frequencies are presented in terms of the non-dimensional parameter Ω given by

$$\Omega = \omega a^2 \sqrt{m_p t_p / D}, \tag{43}$$

where $D = E_p t_p^3 / 12(1 - \nu^2)$ is the plate flexural rigidity, t_p is the plate thickness, m_p is the mass density, ω the natural frequency in rad/s and a is the side length of plate. It can be seen that both values are in good agreement.

3.1.2. *Example* 2

The plate is simply supported with one eccentric stiffener spanning the plate along the centerline as shown in Figure 3. The moduli of elasticity of the plate and the stiffener are $E_p = E_{st} = 30 \times 10^6$ psi ($2{\cdot}07 \times 10^5$ N/mm^2), and the mass densities are $m_p = m_{st}^{xi} = m_{st}^{yi} = 0{\cdot}28$ lb/in^3 ($7{\cdot}83 \times 10^{-6}$ kg/mm^3). The dimensions of the plate are $a = 16$ in (410 mm), $b = 24$ in (600 mm) and the thickness $t_p = 0{\cdot}25$ in (6·33 mm). The stiffener depth $h^{x1} = 0{\cdot}875$ in (22·22 mm) and the thickness $t_s^{xi} = 0{\cdot}5$ in (12·7 mm). This panel was analyzed by Mukherjee and Mukhopadhy [7] using their finite element formulation, by Asku [3] using a finite difference formulation and by Harik and Guo [8] using the finite element method. The panel was analyzed by SQP with 20 variables: 16 out-of-plane and four in-plane design variables. The lowest natural frequency for the structure is shown in Table 2 and as can be seen they are in reasonable agreement.

3.1.3. *Example* 3

The plate is simply supported with concentric stiffening as shown in Figure 4. The plate aspect ratio $\beta = 1$ and the depth and the thickness of the stiffeners is $h_s^{xi} = 4$, $t_s^{xi} = t_p = 1$. The panel was analyzed for two cases; (1) four longitudinal stiffeners $sp^{x1} = sp^{x2} = sp^{x3} = 24$ and (2) five longitudinal stiffeners, $sp^{x1} = sp^{x2} = sp^{x3} = sp^{x4} = 24$. In

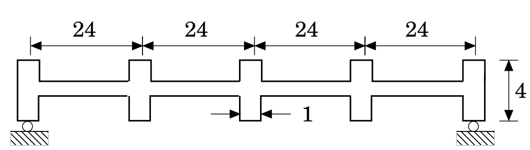

Figure 4. Geometric details of plate with five longitudinal concentric stiffeners.

TABLE 3

Comparison of Ω obtained using SQP and reference [1]

$NS^{x}1$	EI^{xi}/Db	A^{xi}/t_pb	Reference [1]	SQP
3	0·806	0·056	2·6	2·55
4	0·605	0·042	1·464	1·436

Table 3 a comparison is made between the numerical values obtained using SQP and those of reference [1] for the natural frequency parameter Ω which is given by

$$\Omega = (a/(NS^{x} - 1))^{2}\omega\sqrt{m_p t_p/D}, \tag{44}$$

where $(NS^{x} - 1)$ represents the number of bays. It can be seen that the values of reference [21] and the ones obtained using SQP are in good agreement.

3.1.4. *Example* 4

The last verification example is a 2×2 bay continuous plate shown in Figure 5. The line supports are placed at the locations $sp^{x1} = sp^{y1} = 0{\cdot}5$. This plate was analyzed by using the finite strip method by Wu and Cheung [18] and by Koko [9] using a refined finite element formulation. He modelled the structure by four plate elements, one for each bay with a total of 154 degrees of freedom. The natural frequency computed in terms of the non-dimensional parameter Ω is given by

$$\Omega = \omega L^{2}\sqrt{m_p t_p/D} \tag{45}$$

The structure was analyzed by SQP with 20 design variables and the lowest natural frequencies are compared in Table 4 with the values of references [9, 18].

3.2. INFLUENCE OF PLATE/STIFFENER PROPORTIONS

The objectives in this section are to study the influence of plate/stiffener proportions on the lowest natural frequency of the assembly. Specific attention is given to the influence of parameters of the plate and the stiffeners on the natural frequency of the assembled structure. The behaviour, in this section, is detailed for four example panels containing: (1) one longitudinal stiffener, (2) two longitudinal stiffeners, (3) two longitudinal and one transverse stiffeners, and (4) three longitudinal stiffeners. The stiffener profile is chosen to be rectangular. For eccentric stiffening, the values of the parameters γ_{1-4}, δ_1^{xi} and δ_2^{xi} of equations (42) for this profile become

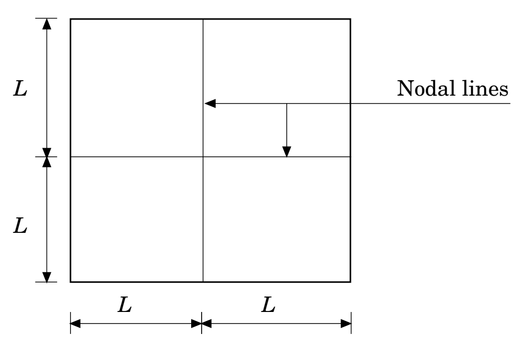

Figure 5. Geometric details of 2×2 continuous plate.

TABLE 4

Comparison of Ω obtained using SQP and references [9, 18]

Figure	Reference	Ω
5	9	20·05
	18	19·74
	Present	19·6

$$\begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \delta_1^{xi} \\ \delta_2^{xi} \end{Bmatrix} = 12(1-\nu^2)\frac{E_{st}}{E_p}\begin{Bmatrix} (b/t_p)(h^{xi}/t_p)(t_s^{xi}/t_p) \\ (h^{xi}/b)(t_s^{xi}/t_p)[4/3(h^{xi}/t_p)^2 + 2(h^{xi}/t_p) + 1] \\ (h^{xi}/t_p)(t_s^{xi}/t_p)[1 + h^{xi}/t_p]^2 \\ (1/6(1+\nu))(h^{xi}/b)(t_s^{xi}/t_p)^3 \\ (1/12(1-\nu^2))(h^{xi}/b)(t_s^{xi}/t_p) \\ \dfrac{1}{48(1-\nu^2)}\dfrac{h^{xi}}{a}\dfrac{t_s^{xi}}{a}\left[\dfrac{1}{3}\dfrac{h^{xi}}{a}\dfrac{t_s^{xi}}{a}\left(\left[\dfrac{h^{xi}}{t_s^{xi}}\right]^2 + 1\right) + \dfrac{t_p}{a}\left(1 + \dfrac{h^{xi}}{t_p}\right)^2\right] \end{Bmatrix}, \tag{46}$$

where a, b and t_p are the length, width and thickness of the plate, and h^{xi}, t_s^{xi} are the depth and the thickness of stiffeners along the x-axis. The values for the concentric configuration can be obtained by similar substitution. Also in the present analysis, the b/t_p ratio is chosen to be 150. Moreover, the $A^{xi}/bt_p = A^{yi}/bt_p$ ratios are kept to 0·1 and the plate aspect ratio at $\beta = 1$ and the warping stiffness of the stiffeners are ignored for all cases.

Figure 6(a) shows the natural frequency parameter, Ω, versus the h^{x1}/t_p ratio for a centrally stiffened panel. The solid curve represents the eccentric while the dashed curve represents the concentric configuration. The solid circles are the numerical values obtained from the analysis using SQP. Starting with an unstiffened plate with unit aspect ratio, the natural frequency parameter Ω equals 19·73, and the mode shape is a half-sine wave. Increasing h^{x1}/t_p, while setting the contribution of the torsional strain energy GJ^{x1}/Db to zero, the natural frequency parameter, Ω, increases until it attains a constant value of 49·3 at $h^{x1}/t_p = 6{\cdot}1$ for the eccentric and 13·7 for the concentric configuration. At this stage, the stiffener subdivides the plate into two sub-panels and freely rotates, since $GJ^{x1}/Db = 0$. The

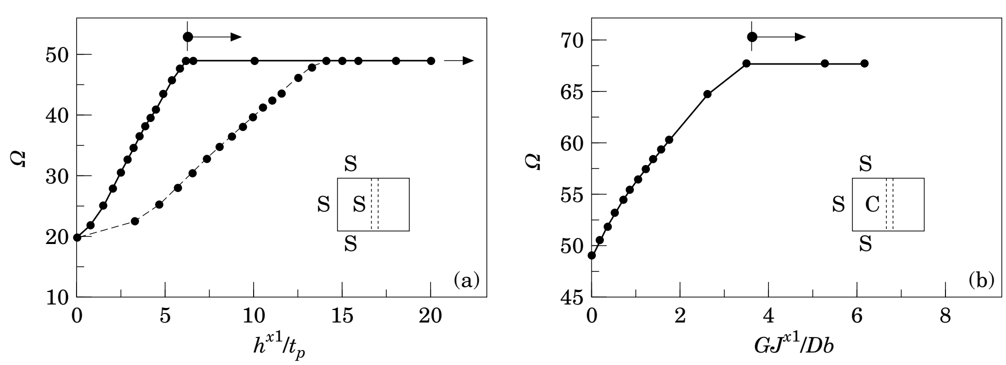

Figure 6. (a) Variation of Ω with h^{x1}/t_p for a plate with one longitudinal stiffener; (b) variation of Ω with GJ^{x1}/Db ratio.

EI^{x1}/Db ratio for this h^{x1}/t_p value is about 17. The ratio h^{x1}/t_p for the concentric configuration is larger since the axis of bending of the plate and the stiffener in this case coincides while, for the eccentric configuration, the value of the inertia of the stiffener is larger. The natural frequency parameter Ω at this stage equals 49·3 which can also be obtained by replacing the parameter β_{sub} by β in equation (14), i.e,.

$$\omega_{sub} = \pi^2(i^2 + j^2\beta_{sub}^2)(1/a^2)\sqrt{D/m_p t_p} = \pi^2(1 + 1 * 4)(1/a^2)\sqrt{D/m_p t_p}$$
$$= 49{\cdot}3(1/a^2)\sqrt{D/m_p t_p} = \omega_{PL}. \quad (47)$$

The first quantity of the above equation is the natural frequency parameter of the sub-panel i.e., a simply supported plate along the four edges of width sp^{x1} and β_{sub} is the aspect ratio of the sub-panel (a/sp^{x1}). Therefore, these values of h^{x1}/t_p are the points where the natural frequencies of the plate and the sub-panels coincide and hence represent the optimum values.

Now, by fixing the EI^{x1}/Db or h^{x1}/t_p ratio at any value along the constant $\Omega = 49{\cdot}3$ line and increasing the torsional stiffness parameter GJ^{x1}/Db of the stiffener, a further increase in the natural frequency parameter Ω can be obtained as shown in Figure 6(b). Note that by fixing h^{x1}/t_p, t_s^{x1}/t_p needs to be increased to increase the J^{x1} value of the stiffener. The stiffener, at this stage, partially restrains the plate against rotation along $b/2$ until it clamps the plate along this side and the Ω value becomes constant at 68 for $GJ^{x1}/Db \approx 3{\cdot}8$. This corresponds to the natural frequency of a plate with three simply supported edges and clamped along the fourth longitudinal edge. Any further increase in t_s^{x1}/t_p produces no further increase in the natural frequency of the plate.

To give numerical insight to the advantage of stiffened plates, assume that the modulus of elasticity of the plate and the stiffener are $E_p = E_{st} = 30 \times 10^6$ psi ($2{\cdot}07 \times 10^5$ N/mm^2), the mass densities $m_p = m_{st}^{xi} = m_{st}^{yi} = 0{\cdot}28$ lb/in^3 ($7{\cdot}83 \times 10^{-6}$ kg/mm^3) and the dimensions of the plate are $a = 30$ in (762 mm), $b = 30$ in (762 mm). If one further assumes that the natural frequency to be attained is $\omega = 266$ Hz, for the unstiffened plate, the required plate thickness to achieve this natural frequency is $t_p = 0{\cdot}5$ in (12·7 mm) and thus the total volume of material required is 450 in^3 ($7{\cdot}4 \times 10^6$ mm^3). By adding a stiffener along the centerline of the plate, this natural frequency can be attained at $h^{x1} = 1{\cdot}22$ in (30·5 mm), $t_s^{x1} = 0{\cdot}5$ in (12·7 mm) and a reduced plate thickness t_p of 0·2 in (5·1 mm). Thus the total volume of the stiffened plate is 198 in^3 ($3{\cdot}24 \times 10^6$ mm^3). Therefore, for the same natural frequency, the stiffened plate requires less than one half the material the unstiffened plate requires, noting that the weight of the stiffener constitutes about 10% of the total weight of the structure.

Figure 7 illustrates the variation of the stiffener spacing with the natural frequency parameter Ω for $0 < (sp^{x1}/b) < 1$ for the eccentric configuration for $EI^{x1}/Db = 20$, i.e., $h^{x1}/t_p = 6{\cdot}65$. Note that this value of h^{x1}/t_p is along the constant Ω line of Figure 6. The incremental value of sp^{x1} was $b/10$. The value of GJ^{x1}/Db is taken to be zero for this case. Note that the curve is symmetric about the centerline, $sp^{x1} = b/2$. As can be seen, the best location for the stiffener, which produces the highest Ω value, is at the centerline of the plate. A possible explanation for this is that when the stiffener is off center, the plate is divided into two sub-panels with different widths, sp^{x1} and sp^{x2}, as shown in Figure 1. The critical one will correspond to the larger sp^{x1} since it will produce the smallest Ω value. If, for example, the stiffener is at $sp^{x1} = 0{\cdot}4b$, the panel with $sp^{x2} = 0{\cdot}6b$ will have a lower natural frequency since it has a lower aspect ratio. Therefore, the largest Ω can be obtained when the natural frequency of both sub-panels coincides, i.e., $sp^{x1} = sp^{x2} = 0{\cdot}5$.

When considering two equally spaced longitudinal stiffeners, with $h^{x1}/t_p = h^{x2}/t_p$ and eccentric configuration, the optimum stiffener slenderness is increased to 10·8 as shown

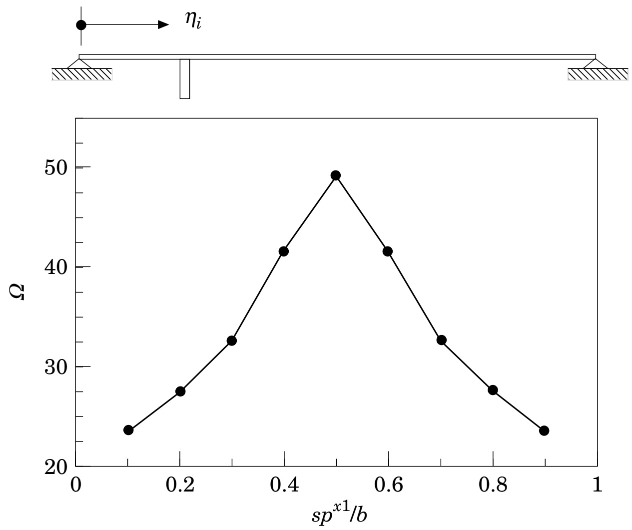

Figure 7. Variation of Ω with the stiffener spacing, sp^{x1}/b.

in Figure 8. Note that since the parameters h^{x1}/t_p and h^{x2}/t_p are increased equally, they are denoted by h^x/t_p in the graph. At this value of h^x/t_p, the stiffeners sub-divide the plate into three sub-panels each of length a and width $b/3$. The maximum Ω value at this stage, ignoring the contribution of the torsional strain energy, is 98·6. This value can also be obtained by replacing the plate parameters, β_{sub} and a_{sub} by β and a in equation (14), i.e.,

$$\omega_{sub} = \pi^2(i^2 + j^2\beta_{sub}^2)\frac{1}{a_{sub}^2}\sqrt{\frac{D}{m_p t_p}} = \pi^2(1 + [3]^2)\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = 98{\cdot}6\,\frac{1}{a^2}\sqrt{\frac{D}{m_p t_p}} = \omega_{PL}, \quad (48)$$

noting that in this case $\beta_{sub} = 3\beta$. It can be seen that the value of the maximum natural frequency parameter is almost doubled by adding an additional transverse stiffener.

By adding a third transverse stiffener along the centerline of the plate, i.e., $sp^{y1} = a/2$, and increasing h^x/t_p and h^y/t_p equally, the optimum h/t_p for the longitudinal and transverse stiffeners is increased to 10·5 for the eccentric and to 21 for the concentric configuration as shown in Figure 9. The natural frequency parameter of the structure, in this case,

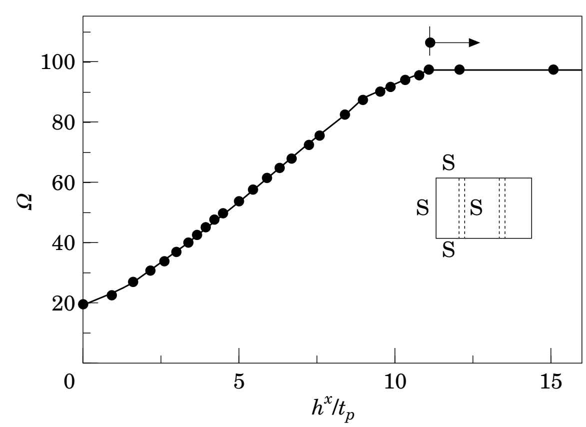

Figure 8. Variation of Ω with h^x/t_p for a plate with two longitudinal stiffeners.

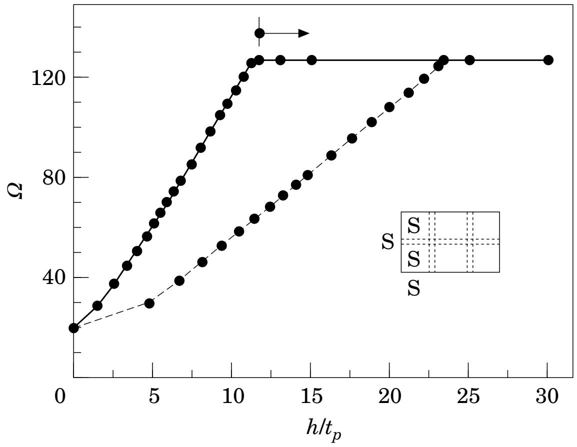

Figure 9. Variation of Ω with h/t_p for a plate with two longitudinal and one transverse stiffeners with eccentric and concentric configurations.

increases to 128·3. At this h^x/t_p value, the stiffeners subdivide the plate into six sub-panels, each with $\beta_{sub} = 1{\cdot}5$. This value of Ω can be obtained by replacing, in equation (14), β_{sub} and a_{sub} by β and a to obtain

$$\omega_{sub} = \pi^2(i^2 + j^2\beta_{sub}^2)\frac{1}{a_{sub}^2}\sqrt{\frac{D}{m_p t_p}} = \pi^2(1 + 1 * [\tfrac{3}{2}]^2)\frac{1}{(a/2)^2}\sqrt{\frac{D}{m_p t_p}} = 128{\cdot}3\sqrt{\frac{D}{m_p t_p}} = \omega_{PL}. \tag{49}$$

If the transverse stiffener is added to in the longitudinal direction instead of the transverse direction, the maximum value of natural frequency parameter Ω of the structure is 167·7, i.e., higher by about 30% from the previous configuration, as shown in Figure 10. The amount of h^x/t_p to subdivide the plate into four equally sub-panels on the other hand is 16·1 for the eccentric configuration.

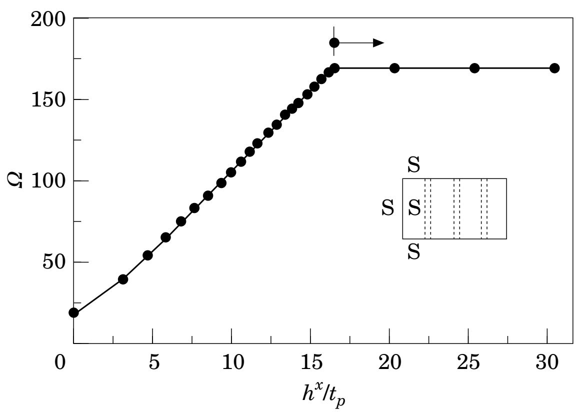

Figure 10. Variation of Ω with h^x/t_p for a plate with three longitudinal stiffeners.

4. CONCLUSIONS

The paper has presented a novel approach for the dynamic analysis of multi-stiffened plates. The structure is idealized as assembled plate and beam elements rigidly connected at their junctions. The strain energy functions of the plate and the stiffener elements are transformed into an unconstrained optimization problem. Sequential quadratic programming is then employed to find the lowest natural frequency for the assembled structure. The method takes into account the discrete nature of the structure, flexible for handling a non-uniform stiffening system and very efficient for linear and non-linear vibration problems. The influence of plate-stiffener proportions on the natural frequency is then detailed for several stiffening configurations. From these graphs it is now possible to determine the finite values of h/t_p which maximize the lowest natural frequency of the structure.

ACKNOWLEDGMENT

The financial support provided to the author from the Natural Sciences and Engineering Council of Canada (NSERC) is gratefully acknowledged.

REFERENCES

1. T. Wah 1964 *Aeronautical Quarterly* **15**, 285–298. Vibration of stiffened plates.
2. G. Asku and R. Ali 1976 *Journal of Sound and Vibration* **48**, 15–25. Free vibration analysis of stiffened plates using finite difference method.
3. G. Asku 1982 *Journal of Applied Mechanics* **49**, 206–212. Free vibration analysis of stiffened plates by including the effect of in-plane inertia.
4. M. Mukhopadhyay 1989 *Journal of Sound and Vibration* **130**, 27–39. Vibration and stability of stiffened plates by semi-analytic finite difference method, part I: consideration of bending displacement only.
5. M. Mukhopadhyay 1989 *Journal of Sound and Vibration* **130**, 41–53. Vibration and stability of stiffened plates by semi-analytic finite difference method, part II: consideration of bending and axial displacements.
6. A. Mukherjee and M. Mukhopadhyay 1986 *Aeronautical Journal* **120**, 267–273. Finite element free vibration analysis of stiffened plates.
7. A. Mukherjee and M. Mukhopadhyay 1989 *Computers and Structures* **33**, 295–305. Finite element free vibration of eccentrically stiffened plates.
8. I. E. Harik and M. Guo 1993 *Computers and Structures* **49**, 1007–1015. Finite element analysis of eccentrically stiffened plates in free vibration.
9. T. S. Koko *Ph.D. Dissertation*, *University of British Columbia*. Super finite elements for non-linear static and dynamic analysis of stiffened plate structures.
10. P. C. Shen, H. Dade and Z. Wang 1987 *Computers and Structures* **27**, 73–78. Static, vibration and stability analysis of stiffened plates using B-spline functions.
11. C. L. Kirk 1970 *Journal of Sound and Vibration* **13**, 375–388. Natural frequency of stiffened rectangular plates.
12. C. L. Kirk 1961 *Journal of the Royal Aerospace Society* **65**, 695–697. Vibration of centrally stiffened rectangular plates.
13. B. R. Long 1960 *Shock and Vibration Bulletin* **38**, 45–53. Vibration of eccentrically stiffened plates.
14. B. R. Long 1971 *Journal of Sound and Vibration* **16**, 323–335. A stiffness-type analysis of the vibration of a class of stiffened plates.
15. D. J. Mead, D. C. Zhu and N. S. Bardell 1988 *Journal of Sound and Vibration* **127**, 19–48. Free vibration of orthogonally stiffened flat plate.
16. P. E. Gill, W. Murray and M. H. Wright 1981 *Practical Optimization*. London: Academic Press Limited.
17. A. W. Leissa 1973 *Journal of Sound and Vibration* **31**, 257–293. The free vibration of rectangular plates.
18. C. L. Wu and Y. K. Cheung 1974 *Earthquake Engineering and Structural Dynamics* **3**, 3–14. Frequency analysis of rectangular plates continuous in one or two directions.

APPENDIX: LIST OF SYMBOLS

A_p	area of the plate per unit width
A^{xi}, A^{yi}	areas of ith stiffener along the x- and y-axes
a, b	length and width of the plate
E_p, E_{st}	Young's moduli for the plate and the stiffeners
e_x	eccentricity of the stiffeners
$F_i(\xi)$, $G_j(\eta)$	out-of-plane displacement functions
$B_m(\xi)$, $D_n(\eta)$	in-plane displacement functions
$E_r(\xi)$, $H_s(\eta)$	in-plane displacement functions
G	shear modulus
h^{xi}, h^{yi}	depth of ith stiffener along the x- and y-axes
I^p	moment of inertia of the plate per unit width
I_y^{xi}, I_x^{yi}	second moment of inertia about the major axis of typical x- and y-stiffener
I_z^{xi}, I_z^{yi}	second moment of inertia, about the minor axis, of typical x- and y-stiffener
I_0^{xi}, I_0^{yi}	polar moment of inertia of typical x- and y-stiffener
J^{xi}, J^{yi}	torsional rigidity of typical x- and y-stiffener
M_{xx}, M_{yy}, M_{xy}	results moment components
m_p	mass density of the plate
m_{st}^{xi}, m_{st}^{yi}	mass densities of typical x- and y-stiffeners
N_{xx}, N_{yy}, N_{xy}	resultant force components
Q^{xi}, Q^{yi}	first moment of inertia of typical x- and y-stiffener
W, U, V	out-of-plane and in-plane displacements
SC	support condition
sp^{xi}, sp^{yi}	stiffener's spacing in the x- and y-direction
t_p	thickness of the plate
t_s^{xi}, t_s^{yi}	thickness of typical x- and y-stiffener
U_{st}^{xi}, U_{st}^{yi}	total strain energies of stiffeners along the x- and y-axes
β	the plate aspect ratio $= a/b$
ξ, η	non-dimensional parameters $= x/a$ and y/b
ϵ_{ij}^p	strain components of the plate
ϵ_{xx}^{xi}, ϵ_{yy}^{yi}	axial strains of stiffeners along the x- and y-axes
σ_{ij}^p	stress components of the plate
Ω	non-dimensional natural frequency parameter
ω	natural frequency of the structure